Solving Endrings with low memory via isogeny ladders and small orientations

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joint work with Lorenz Panny, Ryan Rueger, Aleksei Udovenko

Technical University of Munich

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AM-PQC Workshop, Ohrid

Finding endomorphism rings: quite hard, right?

Problem: Given $E/\mathbb{F}_{p^2} \in \mathcal{S}_{p^2}$ supersingular elliptic curve, find explicit description of $\mathrm{End}(E)$.

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1. Walk randomly in the supersingular isogeny graph until we find a curve defined over $\mathbb{F}_p$:
   $E \rightsquigarrow E_1 \in \mathcal{S}_p$ (hard, takes $\widetilde{O}(\#\mathcal{S}_{p^2}/\#\mathcal{S}_p)) \subseteq \widetilde{O}(p^{1/2})$)

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- Choice of target subgraph: replace $\mathcal{S}_p$ with other special subgraphs...

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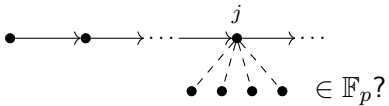
Looking for oriented curves

Neighbourhood inspection from SuperSolver:[2] check whether any ℓ-neighbours are over $\mathbb{F}_p$.



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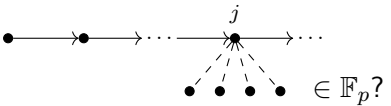

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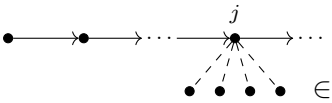

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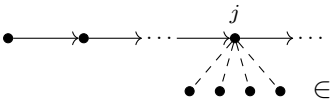

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Extend this idea: Fix a set of quadratic rings, say $\mathcal{O}_d = \mathbb{Z}[\sqrt{-dp}]$ for several d.
For each d, fix $E_{0,d}$ that is $\mathcal{O}_d$-orientable and with known End ring.

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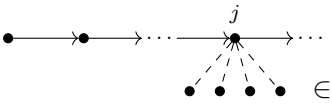

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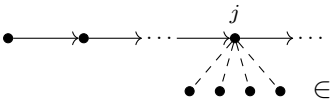

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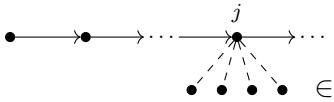

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How to check orientability? For $\mathcal{O} = \mathbb{Z}[\sqrt{-dp}]$ $\rightsquigarrow$ test if E, E^p are d-isogenous.
$\mathcal{O} = \mathbb{Z}[\ell_1\sqrt{-\ell_2 p}]$ $\rightsquigarrow$ can use gcd and resultants, generalizing SuperSolver.[3]

[2] Corte-Real Santos, Costello, Shi. Eprint 2021/1488
[3] Corte-Real Santos, Herlédan Le Merdy, Macula, Meyer, Morrison, Orvis. Eprint 2026/1219

Isogeny ladders

$$E_{0,0} \xrightarrow{2} E_{0,1} \xrightarrow{2} \dots \xrightarrow{2} E_{0,n}$$

- Linear walk: each 2-isogeny arrow costs a sqrt.

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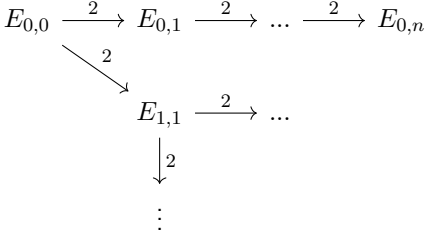

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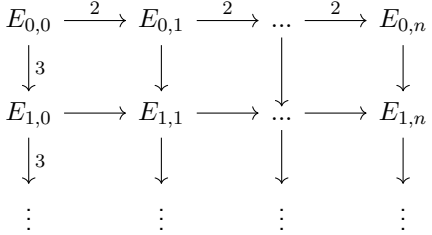

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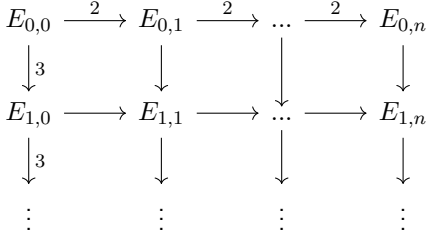

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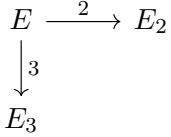

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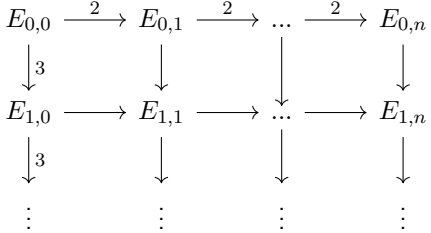

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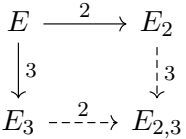

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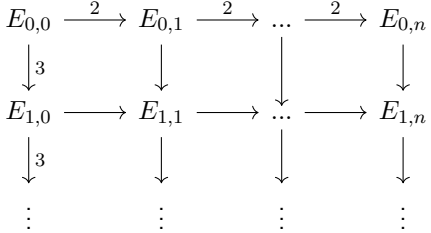

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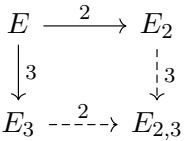

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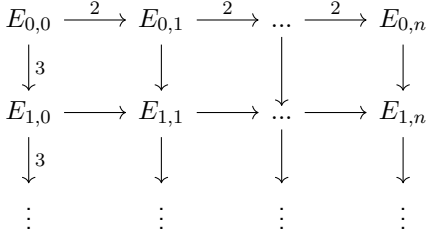

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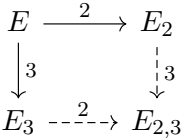

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✓ Square roots required only for the outer sides of the grid
⇝ sqrts amortized out, asymptotic $\log(p)$-speedup

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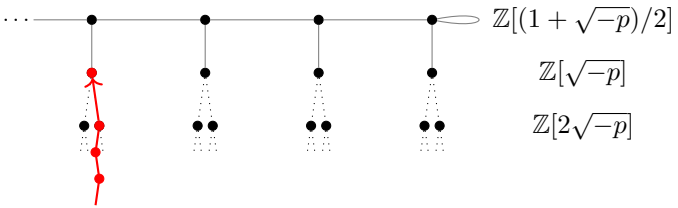

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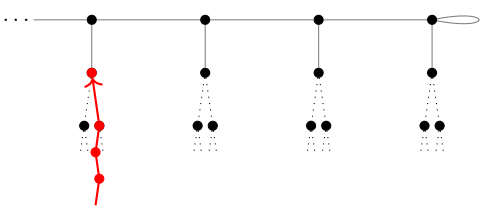

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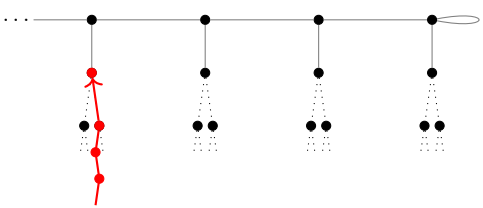

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$\rightsquigarrow$ only part of the target subgraph is "visible" from outside. ⚠

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The shape of the $\mathbb{Z}[\sqrt{-dp}]$-oriented ℓ-isogeny graph depends on

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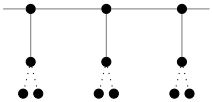
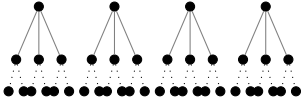
$\ell = 2$			

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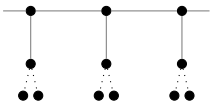
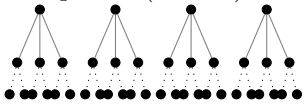
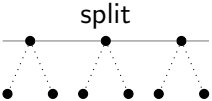
$\ell = 2$			
ℓ odd	split 	ramified 	inert 

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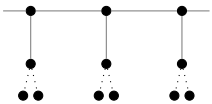
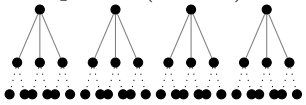
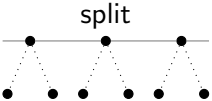
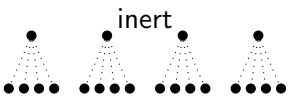
$\ell = 2$			
ℓ odd	split 	ramified 	inert 

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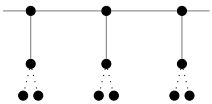
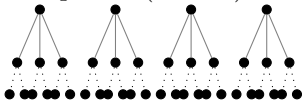
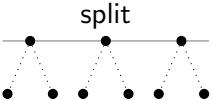
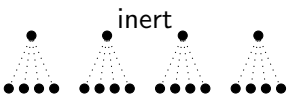
$\ell = 2$			
ℓ odd	split 	ramified 	inert 

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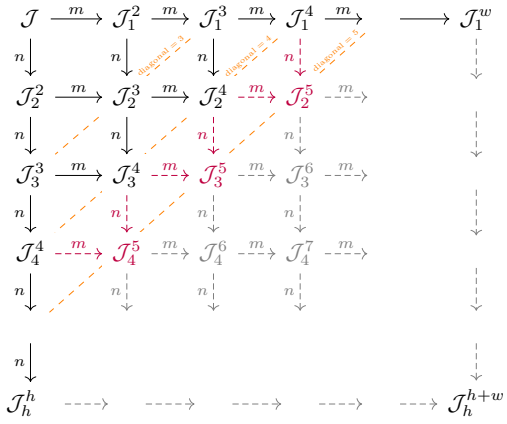
Implementation remarks



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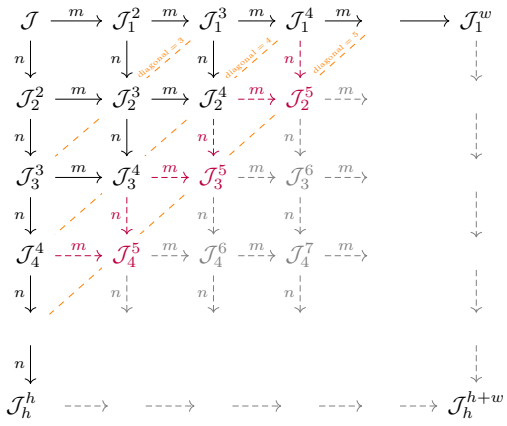

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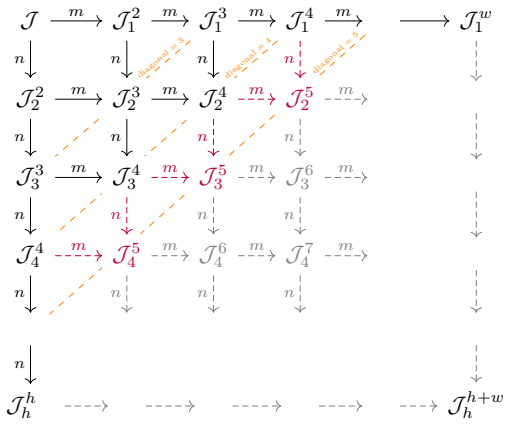

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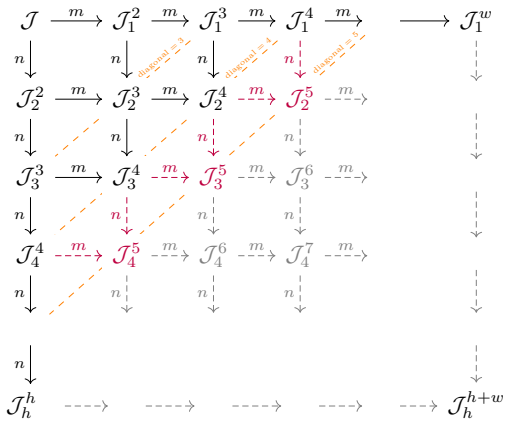

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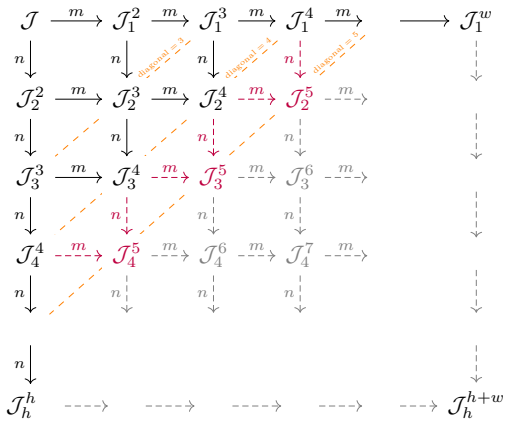

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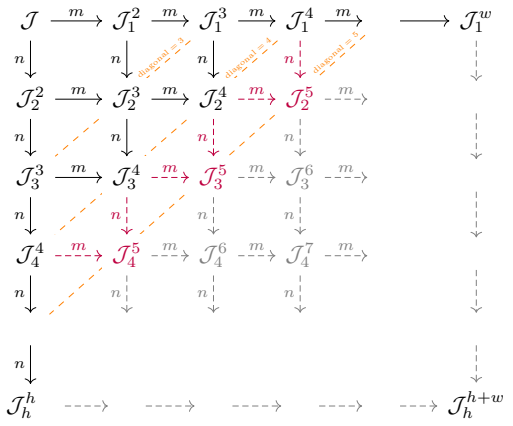

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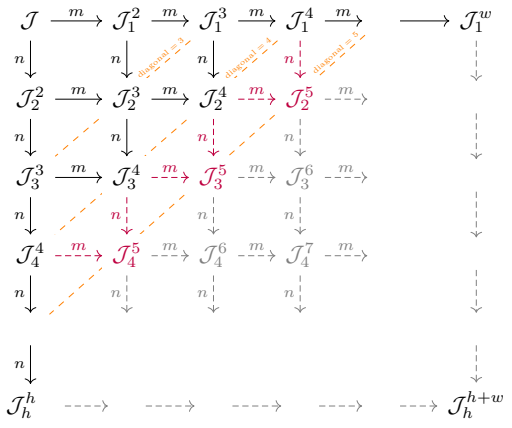

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